A normally supercompact Parovičenko space

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These results are from joint paper with W. Kubiś and S. Turek. We prove an analog of the Parovičenko Theorem in the category consisting of normally supercompact spaces and continuous maps preserving convexity.

The following properties of the remainder $P = \omega^*$ are well-know:

1. P is a 0-dimensional compact space without isolated points with weight 2^ω
2. every two disjoint open F_σ in P have disjoint closure
3. every non-empty G_δ set in P has a non-empty interior

Theorem

For every compact space P the following conditions are equivalent

- *P is satisfying conditions (1)-(3)*
- *for every continuous mapping f of P onto a compact metrizable space X and every continuous mapping g of a compact metrizable space Y onto X there exists a continuous mapping h of P onto Y such that $g \circ h = f$*

P is a Parovičenko space

$$\begin{tikzcd} P \arrow[d, two heads, "f"'] \arrow[dr, dashed, two heads, "h"] & \\ X & Y \arrow[l, two heads, "g"'] \end{tikzcd}$$

where $\mathrm{w}(X), \mathrm{w}(Y) < \mathrm{w}(P)$

A family $\mathcal{F}$ is *linked* if $A \cap B \neq \emptyset$ for every $A, B \in \mathcal{F}$.

A closed subbase $\mathcal{F}$ is *normal subbase* if for any $S, T \in \mathcal{F}$ with $S \cap T = \emptyset$ there are $S', T' \in \mathcal{F}$ such that $S' \cap T = \emptyset = T' \cap S$ and $T' \cup S' = X$.

A topological space is *supercompact* if it has a subbase $\mathcal{B}$ for its closed sets, such that every linked family $\mathcal{F} \subseteq \mathcal{B}$ has nonempty intersection.

A topological space X which possesses a binary normal subbase is called *normally supercompact*.

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Normally supercompact spaces, have geometric aspect as generally understood convexity.

Recall that *interval convexity* (*convexity* for short) on the set X is the family $\mathcal{G}$ of subsets of X which satisfies the following conditions:

1. $\emptyset, X \in \mathcal{G}$;
2. if $\mathcal{G}' \subseteq \mathcal{G}$ then $\bigcap \mathcal{G}' \in \mathcal{G}$;
3. if $A \subseteq X$ and for each $a, b \in A$ there is $C \in \mathcal{G}$ with $a, b \in C \subseteq A$ then $A \in \mathcal{G}$.

Elements of $\mathcal{G}$ are called convex sets. A convex set with convex completion is called *halfspace*.

The map $f: X \to Y$, is called *convexity preserving map* (*cp-map* for short), if $f^{-1}(G) \in \mathcal{F}$ for each $G \in \mathcal{G}$, where X and Y are spaces with convexities $\mathcal{F}$ and $\mathcal{G}$ respectively.

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Let X be a normally supercompact space with normal binary subbase $\mathcal{B}$. Let define an interval map $I_{\mathcal{B}}\colon X \times X \to \mathcal{P}(X)$ by the formula:

$$I_{\mathcal{B}}(a,b) = \bigcap\{B \in \mathcal{B}\colon a,b \in B\}.$$

The family $\mathcal{G}_{I_{\mathcal{B}}} = \{A \subseteq X\colon I_{\mathcal{B}}(a,b) \subseteq A \text{ for each } a,b \in A\}$
is a convexity containing $\mathcal{B}$

We will consider category $\mathfrak{CM}$ consisting of all normally supercompact spaces and surjective continuous cp-maps.

Such maps will be called *epimorphisms* and denoted $f : X \twoheadrightarrow Y$ for $X, Y \in \mathfrak{CM}$.

Let κ be an infinite regular cardinal.

A normally supercompact space P is called *κ-Parovičenko* if for any epimorphisms $q\colon P \twoheadrightarrow L$ and $f\colon K \twoheadrightarrow L$, where K, L are normally supercompact space of weight $< \kappa$ there exists epimorphism $g\colon P \twoheadrightarrow K$ such that $f \circ g = q$.

$$\begin{tikzcd}
P \arrow[d, two heads, "q"'] \arrow[dr, dashed, two heads, "g"] & \\
L & K \arrow[l, two heads, "f"']
\end{tikzcd}$$

Theorem (W. Kubiś, A. K., S. Turek)

Every κ-Parovičenko space is zero-dimensional.

Let P be a normally supercompact space. A collection of all clopen halfspace in P we denote by $H(P)$ and $H(P)^+ = H(P) \setminus \{\emptyset\}$. The following results are obtained:

Theorem (W. Kubiś, A. K., S. Turek)

There exists a κ-Parovičenko normally supercompact space P of the weight $\leq \kappa^{<\kappa}$.

Theorem (W. Kubiś, A. K., S. Turek)

Let P be a κ-Parovičenko space. Then:

1. *Every normally supercompact space of weight $\leq \kappa$ is a homomorphic image of P.*
2. *If $w(P) = \kappa$ then for each epimorphisms $f, g \colon P \twoheadrightarrow K$, where $w(K) < \kappa$, there is an isomorphism $h \colon P \to P$ such that $f = g \circ h$.*
3. *There exists at most one (up to isomorphism) κ-Parovičenko space of weight κ.*

Theorem (W. Kubiś, A. K., S. Turek)

Given a normally supercompact space P, the following conditions are equivalent:

(a) *P is κ-Parovičenko space,*

(b) *if $f : P \to K$ is an epimorphism, $w(K) < \kappa$ and $E, F \subseteq K$ are closed convex sets such that $E \cup F = K$ then there exists $H \in H(P)$ such that $f[H] = E$ and $f[P \setminus H] = F$,*

(c) *P is zero-dimensional and satisfies conditions:*

  (M1) *If $\mathcal{A}, \mathcal{B} \subseteq H(P)$, $|\mathcal{A} \cup \mathcal{B}| < \kappa$ and any $A \in \mathcal{A}$ and $B \in \mathcal{B}$ are disjoint then there exists $C \in H(P)$ with $\bigcup \mathcal{A} \subseteq C$ and $\bigcup \mathcal{B} \subseteq P \setminus C$,*

  (M2) *If $\mathcal{A} \subseteq H(P)$ is linked and $|\mathcal{A}| < \kappa$, then there exists $C \in H(P)^+$ such that $C \subseteq \bigcap \mathcal{A}$*

  (M3) *for each $A \in H(P)^+$ there exists $B_0, B_1 \in H(P)^+$ such that $B_0 \cap B_1 = \emptyset$ and $B_0 \cup B_1 \subseteq A$.*